

Continuity and Stability of Partial Entropic Sums

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Extensions of Fannes' inequality with partial sums of the Tsallis entropy are obtained for both the classical and quantum cases. The definition of k -th partial sum under the prescribed order of terms is given. Basic properties of introduced entropic measures and some applications are discussed. The derived estimates provide a complete characterization of the continuity and stability properties in the refined scale. The results are also reformulated in terms of Uhlmann's partial fidelities.

Keywords: continuity, stability, Tsallis entropy, Fannes' inequality, Ky Fan's norm

I. INTRODUCTION

The concept of entropy is much widely used in statistical mechanics, quantum physics and information theory. In the last decades, new direction of researches has been formed on the joining point of these fields. The discipline is usually referred to as "quantum information" [11, 14]. In all the topics, it is necessary to quantify the informational contents of a state by some functionals. For these purposes, various entropic quantities have been utilized [12, 14]. The Shannon entropy for probability distributions and the von Neumann entropy for density operators are widely adopted measures. Many other functionals like the Tsallis entropies [9, 20], the Rényi entropies [19], and the quasi-entropies [13] have been recognized as needed in specialized topics. Quantum entropic measures are nontrivial mathematical subjects, and active study of them has led to some deep insights (see, e.g., historical remarks in [5]).

Any entropic measure must have good functional properties. Fannes showed the continuity property for the von Neumann entropy [7]. Fannes' inequality gives an upper bound on a potential change of the entropy when the quantum state is altered. Note that the dimension of the state space is explicitly involved in the bound. Using the classical Fano inequality, Fannes' bound has been sharpened (see theorem 3.8 in [14]). Recently, its analogs for Tsallis entropies were obtained [8, 23]. Further, the stability property is very important, mainly due to its direct relation to observability [1]. In spite of direct formal relation between the Tsallis α -entropy and the Rényi α -entropy, the former is stable [1, 23], while the latter is not stable for $\alpha \neq 1$ [1, 10]. There are other important properties, for example, related to nonextensive additivity. The quantum Tsallis entropy of order $\alpha > 1$ is subadditive (that has been conjectured in [15] and later proved in [3]). Concerning the minimum relative-entropy principle, an extension of triangle inequality to the nonextensive case was developed [2]. In the present paper, we focus an attention on the continuity and stability properties.

Functional properties may be asked with respect to separate components in expression for entropy. A study of partial sums of some measure may be useful in various regards. By means of the Ky Fan norms, many important results can be extended to the class of unitarily invariant norms. For instance, all the metrics induced by unitarily invariant norms are tantamount in posing the cryptographic exponential indistinguishability [18]. The equivalence of pairs of density operators has been resolved in terms of Uhlmann's partial fidelities [22]. In view of very importance of both the continuity and stability, one of possible development of the issue is to examine these properties for partial entropic sums. The paper is organized as follows. In Section II, we deal with the classical case and review auxiliary material. In Section III, the case of density operators with some applications is studied. In Sections IV and V the main result will be established for two different ranges of parameter α . In an explicit form, the bounds on differences between k -th partial entropic sums are dependent on integer k only. In this respect, the obtained inequalities differ from Fannes' inequality. We also show that examined partial sums are stable in the sense inspired by Lesche [10].

II. DEFINITIONS FOR CLASSICAL DISTRIBUTIONS

Let $\mathbf{x} = (x_1, \dots, x_m)$ be an element of real space \mathbb{R}^m . For $k = 1, \dots, m$, we define the function

$$G_{(k)}(\mathbf{x}) := \sum_{i=1}^k |x_i|^{\downarrow}, \quad (2.1)$$

where the arrows down indicate that absolute values are put in the decreasing order. Note that $G_{(m)}(\mathbf{x})$ poses l_1 -norm and $G_{(1)}(\mathbf{x})$ poses l_∞ -norm on \mathbb{C}^m [4]. In terms of these symmetric gauge functions, the Ky Fan norms of operators are defined. Let $x \mapsto f(x)$ be a function of real variable. For any $f : [0; 1] \rightarrow \mathbb{R}_+ \equiv [0; +\infty)$, we introduce the map

$\mathbf{x} \mapsto G_{(k)}[f(\mathbf{x})]$ by

$$G_{(k)}[f(\mathbf{x})] := \sum_{i=1}^k |f(x_i)|^\downarrow. \quad (2.2)$$

Here a vector \mathbf{x} is assumed to be contained in the probability simplex Δ_m of those vectors that $G_{(m)}(\mathbf{x}) = 1$ and $x_i \geq 0$ for all i . Let us put the α -entropy function [20]

$$\eta_\alpha(x) := -x^\alpha \ln_\alpha(x) = \frac{x^\alpha - x}{1 - \alpha}, \quad (2.3)$$

where the α -logarithmic function $\ln_\alpha(x) \equiv (x^{1-\alpha} - 1)/(1 - \alpha)$ is defined for $\alpha \geq 0$, $\alpha \neq 1$ and $x \geq 0$. The α -logarithmic function converges to $\ln x$ as $\alpha \rightarrow 1$. In the following, we avoid the case $\alpha = 0$ in which we have $\eta_\alpha(0) \neq 0$.

Remark II.1. In the range $x \in [0; 1]$ the function $\eta_\alpha(x)$ is concave, $\eta_\alpha(0) = \eta_\alpha(1) = 0$. Its maximum in this range is reached at the point $x_0(\alpha) = \alpha^{1/(1-\alpha)}$. It can be shown that $dx_0/d\alpha > 0$ for $\alpha \in [0; +\infty)$, and $x_0(1) = 1/e$, $x_0(2) = 1/2$, $x_0(+\infty) = 1$.

Definition II.2. Let $\mathbf{p} = (p_1, \dots, p_m)$ be m -dimensional probability vector. For $k = 1, \dots, m$, the k -th partial entropic sum is defined by

$$H_\alpha^{(k)}(\mathbf{p}) := G_{(k)}[\eta_\alpha(\mathbf{p})]. \quad (2.4)$$

In the case $k = m$, we obtain the entropy itself. This measure is usually referred to as "Tsallis entropy", although it was first studied by Havrda and Charvát [9]. The Tsallis entropy has many applications in nonextensive statistical mechanics. Unlike the Shannon entropy and the Rényi entropy, the Tsallis entropy is not additive [3, 15]. The partial sums of Tsallis entropy enjoy a few obvious properties.

(1C) *Positivity:* $H_\alpha^{(k)}(\mathbf{p}) \geq 0$; $H_\alpha^{(k)}(\mathbf{p}) = 0$ only in corners of the simplex Δ_m .

(2C) *Non-decrease with respect to the order:* if $k < k'$ then $H_\alpha^{(k)}(\mathbf{p}) \leq H_\alpha^{(k')}(\mathbf{p})$.

(3C) *Symmetry:* if \mathbf{q} is obtained by permutation of \mathbf{p} then $H_\alpha^{(k)}(\mathbf{q}) = H_\alpha^{(k)}(\mathbf{p})$.

(4C) *Expansibility:* $H_\alpha^{(k)}(p_1, \dots, p_m, 0) = H_\alpha^{(k)}(p_1, \dots, p_m)$.

(5C) *Product monotonicity:* $H_\alpha^{(k)}(\mathbf{p}) \leq H_\alpha^{(kn)}(\mathbf{p} \times \mathbf{q})$, where $\mathbf{p} \times \mathbf{q}$ denotes the joint probability distribution with elements $p_i q_j$ ($j = 1, \dots, n$).

These properties follow from Definition II.2 at once. Only the last demands some comments. In effect, a stronger statement takes place.

Lemma II.3. Let $\mathbf{r} = \{r_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$ be a probability distribution. Then it holds that

$$H_\alpha^{(k)}(\mathbf{p}) \leq H_\alpha^{(kn)}(\mathbf{r}), \quad (2.5)$$

where $p_i = \sum_{j=1}^n r_{ij}$ are elements of the marginal probability distribution.

Proof. Let \mathcal{A} be k -subset of the set $\{1, \dots, m\}$ such that $H_\alpha^{(k)}(\mathbf{p}) = \sum_{i \in \mathcal{A}} \eta_\alpha(p_i)$. Then we have

$$\begin{aligned} H_\alpha^{(k)}(\mathbf{p}) &= \frac{1}{1-\alpha} \sum_{i \in \mathcal{A}} \left(\sum_{j=1}^n r_{ij} \right)^\alpha - \frac{1}{1-\alpha} \sum_{i \in \mathcal{A}} \sum_{j=1}^n r_{ij} \\ &\leq \frac{1}{1-\alpha} \sum_{i \in \mathcal{A}} \sum_{j=1}^n r_{ij}^\alpha - \frac{1}{1-\alpha} \sum_{i \in \mathcal{A}} \sum_{j=1}^n r_{ij} = \sum_{i \in \mathcal{A}} \sum_{j=1}^n \eta_\alpha(r_{ij}), \end{aligned} \quad (2.6)$$

because $\left(\sum_{j=1}^n r_{ij} \right)^\alpha \leq \sum_{j=1}^n r_{ij}^\alpha$ for $\alpha < 1$, and $\left(\sum_{j=1}^n r_{ij} \right)^\alpha \geq \sum_{j=1}^n r_{ij}^\alpha$ for $\alpha > 1$. In (2.6), the last sum is taken over those pairs (i, j) that $i \in \mathcal{A}$ and $1 \leq j \leq n$. The number of such pairs is kn , whence this sum is not greater than $H_\alpha^{(kn)}(\mathbf{r})$. \square

When m and α are fixed, the maximum of the Tsallis entropy [8, 23]

$$\max \left\{ H_\alpha^{(m)}(\mathbf{p}) : \mathbf{p} \in \Delta_m \right\} = \ln_\alpha(m) \quad (2.7)$$

is reached if and only if $p_i = 1/m$ for all $i = 1, \dots, m$. It is important that this optimal probability vector is independent of α . Conversely, for k -th partial entropic sum $H_\alpha^{(k)}(\mathbf{p})$ the maximizing probability vector is dependent on both k and α . To find explicitly the maximum of k -th partial sum, we should maximize the function $G_{(k)}[\eta_\alpha(\mathbf{x})]$ of vector $\mathbf{x} \in \mathbb{R}^m$ under the conditions $x_i \geq 0$ and $G_{(k)}(\mathbf{x}) \leq 1$. Without equality $G_{(k)}(\mathbf{x}) = 1$, the task becomes difficult since the function $\eta_\alpha(x)$ is not scale-invariant. Nevertheless, we can give simple lower and upper bounds

$$\ln_\alpha(k) \leq \max \left\{ H_\alpha^{(k)}(\mathbf{p}) : \mathbf{p} \in \Delta_m \right\} \leq \ln_\alpha(k+1) \quad (2.8)$$

with high accuracy for sufficiently large k . More precisely, the relative error of estimate by these bounds does not exceed $(k^\alpha \ln_\alpha(k+1))^{-1}$. When $p_i = 1/k$ for $1 \leq i \leq k$ and $p_i = 0$ for $k+1 \leq i \leq m$, k -th partial sum is $\ln_\alpha(k)$, whence the lower bound follows. Further, for each probability vector \mathbf{p} with $G_{(k)}(\mathbf{p}) < 1$ we take another one \mathbf{p}' such that $p'_i = p_i$ for $i = 1, \dots, k$, and $p'_{k+1} = 1 - G_{(k)}(\mathbf{p})$. By construction, we now have $G_{(k)}(\mathbf{p}') = 1$ and

$$H_\alpha^{(k)}(\mathbf{p}) \leq H_\alpha^{(k+1)}(\mathbf{p}') \leq \ln_\alpha(k+1), \quad (2.9)$$

whence the upper bound follows. For $k = 1$, the lower bound in (2.8) becomes trivial. Here we will use the exact expression for the maximum, namely $\eta_\alpha(\alpha^{1/(1-\alpha)})$.

Remark II.4. Since $\alpha^{1/(1-\alpha)} \rightarrow e^{-1}$ as $\alpha \rightarrow 1$, the maximum of second partial sum $H_1^{(2)}$ is equal to $2\eta_1(1/e) = 2/e \approx 0.74$. This is larger than the maximal value $\ln 2 \approx 0.69$ of the binary Shannon entropy. So the left-hand side of (2.8) is lower bound too. For partial sums of the Shannon entropy, the relative error of estimate by (2.8) is almost 0.1 for $k = 5$ and strictly less than 0.1 for $k > 5$.

What is a reason for separate analysis of a partial sums stability? Consider the two-dimensional probability vectors $\mathbf{p} = ((1-\varepsilon)/2, (1+\varepsilon)/2)$ with strictly positive $\varepsilon \ll 1$ and $\mathbf{q} = (1/2, 1/2)$. The $\eta_1(x) = -x \ln x$ is maximal for $x = 1/e$, whence $\eta_1((1-\varepsilon)/2) > \eta_1((1+\varepsilon)/2)$. Putting $\Delta^{(k)} = |H_1^{(k)}(\mathbf{p}) - H_1^{(k)}(\mathbf{q})|$, we write

$$\Delta^{(1)} = \eta_1((1-\varepsilon)/2) - \eta_1(1/2) = (-(1-\varepsilon)\ln(1-\varepsilon) - \varepsilon \ln 2)/2, \quad (2.10)$$

$$\Delta^{(2)} = H_1^{(2)}(\mathbf{q}) - H_1^{(2)}(\mathbf{p}) = ((1+\varepsilon)\ln(1+\varepsilon) + (1-\varepsilon)\ln(1-\varepsilon))/2. \quad (2.11)$$

The ratio of $\Delta^{(1)}$ to $\Delta^{(2)}$ is equal to $(1 - \ln 2)\varepsilon^{-1} + O(1)$ and not bounded as $\varepsilon \rightarrow +0$. Thus, the difference between partial entropic sums can be arbitrarily large in comparison with the difference between the total entropies. We have seen this effect in a vicinity of the uniform distribution. So the continuity and stability of some entropic functional themselves do not imply the same for corresponding partial sums. It turns out, however, that these properties still hold for partial sums of Tsallis' entropy. So we will examine differences between k -th partial sums of Tsallis entropy of two m -dimensional probability vectors \mathbf{p} and \mathbf{q} . It is natural to estimate this difference in terms of partitioned classical distances $G_{(k)}(\mathbf{p} - \mathbf{q})$. These distances and their quantum extensions via Ky Fan's norms were introduced in [17]. As it is shown below, both the classical and quantum cases can similarly be treated.

III. THE CASE OF DENSITY OPERATORS

Let \mathcal{H} be d -dimensional Hilbert space, and let \mathbf{X} be linear operator on \mathcal{H} . By $\text{spec}(\mathbf{X}) \equiv \{\lambda_i(\mathbf{X})\}$ we denote the set of its eigenvalues. For any \mathbf{X} , the operator $\mathbf{X}^*\mathbf{X}$ is positive, i.e. $\langle \psi | \mathbf{X}^*\mathbf{X} | \psi \rangle \geq 0$ for all $|\psi\rangle \in \mathcal{H}$. The operator $|\mathbf{X}|$ is defined as a unique positive square root of $\mathbf{X}^*\mathbf{X}$. The eigenvalues of $|\mathbf{X}|$ counted with multiplicities are the singular values $\sigma_j(\mathbf{X})$ of \mathbf{X} [4]. For $k = 1, \dots, d$, the Ky Fan k -norm is defined as [4]

$$\|\mathbf{X}\|_{(k)} := G_{(k)}(\sigma(\mathbf{X})) \equiv \sum_{j=1}^k \sigma_j^\downarrow(\mathbf{X}). \quad (3.1)$$

For given $f(x)$, the function of normal operator $\mathbf{X} = \sum_{i=1}^d \lambda_i |e_i\rangle\langle e_i|$ is introduced by

$$f(\mathbf{X}) = \sum_{i=1}^d f(\lambda_i) |e_i\rangle\langle e_i|, \quad (3.2)$$

where the eigenvectors $|e_i\rangle$ are orthonormal. We also have $\|\mathbf{X}\|_{(k)} = G_{(k)}(\lambda(\mathbf{X}))$ for each normal operator \mathbf{X} .

Definition III.1. Let ρ be density operator on \mathcal{H} . For $k = 1, \dots, d$, the k -th partial entropic sum of quantum state ρ is defined by

$$S_\alpha^{(k)}(\rho) := \|\eta_\alpha(\rho)\|_{(k)}. \quad (3.3)$$

Remark III.2. The eigenvalues $\{p_i\}$ of density operator ρ are naturally treated as probabilities due to their positivity and the normalization $\text{tr}(\rho) = 1$. Because the function $\eta_\alpha(x)$ is nonnegative for $x \in [0; 1]$, the above definition and (3.2) lead to

$$S_\alpha^{(k)}(\rho) = H_\alpha^{(k)}(\mathbf{p}) . \quad (3.4)$$

In the case $k = d$, we have the quantum Tsallis entropy [8, 23]. The quantum partial entropic sums enjoy similar properties to classical ones.

(1Q) *Positivity:* $S_\alpha^{(k)}(\rho) \geq 0$; $S_\alpha^{(k)}(\rho) = 0$ if and only if ρ is projector.

(2Q) *Non-decrease with respect to the order:* if $k < k'$ then $S_\alpha^{(k)}(\rho) \leq S_\alpha^{(k')}(\rho)$.

(3Q) *Symmetry:* if $\text{spec}(\varrho) = \text{spec}(\rho)$ then $S_\alpha^{(k)}(\varrho) = S_\alpha^{(k)}(\rho)$.

(4Q) *Expansibility:* if the Hilbert space \mathcal{H} is extended to $\mathcal{H} \oplus \mathcal{K}$ then $S_\alpha^{(k)}(\rho)$ is not changed for all density operators ρ on \mathcal{H} .

(5Q) *Product monotonicity:* $S_\alpha^{(k)}(\rho) \leq S_\alpha^{(kN)}(\rho \otimes \omega)$, where ω denotes some density operator on N -dimensional Hilbert space.

The entry (3Q) contains the unitary invariance as a particular case. Like (5C), the property (5Q) can be extended to density operators of a specific kind. Suppose a composite 'AB' of systems 'A' and 'B' is described by density operator $\tilde{\rho}$ on the tensor product $\mathcal{H}_A \otimes \mathcal{H}_B$. The reduced density operators for systems 'A' and 'B' are given by

$$\rho_A = \text{tr}_B(\tilde{\rho}) , \quad \rho_B = \text{tr}_A(\tilde{\rho}) , \quad (3.5)$$

where the partial traces are taken over \mathcal{H}_B and \mathcal{H}_A respectively [11, 14]. In the two spectral decompositions

$$\rho_A = \sum_{i=1}^d a_i |i\rangle \langle i| , \quad \rho_B = \sum_{\mu=1}^N b_\mu |\mu\rangle \langle \mu| , \quad (3.6)$$

the $|i\rangle$'s form an orthonormal basis in \mathcal{H}_A , the $|\mu\rangle$'s form an orthonormal basis in \mathcal{H}_B , $\text{spec}(\rho_A) = \{a_i\}$ and $\text{spec}(\rho_B) = \{b_\mu\}$. Then the vectors $|i\mu\rangle \equiv |i\rangle \otimes |\mu\rangle$ form orthonormal basis in the product space $\mathcal{H}_A \otimes \mathcal{H}_B$.

Theorem III.3. *If there hold: (i) the operators $\tilde{\rho}$ and $\rho_A \otimes \rho_B$ are commuting, and (ii) $a_i b_\mu = a_j b_\nu$ only if $i = j$ and $\mu = \nu$, then*

$$S_\alpha^{(k)}(\rho_A) \leq S_\alpha^{(kN)}(\tilde{\rho}) , \quad S_\alpha^{(k)}(\rho_B) \leq S_\alpha^{(kd)}(\tilde{\rho}) . \quad (3.7)$$

Proof. In the fixed basis $\{|i\mu\rangle\}$, the operator $\tilde{\rho}$ can be represented as

$$\tilde{\rho} = \sum_{i\mu} \sum_{j\nu} c(i\mu|j\nu) |i\mu\rangle \langle j\nu| , \quad (3.8)$$

where coefficients $c(i\mu|j\nu) = \langle i\mu|\tilde{\rho}|j\nu\rangle$. Further, the product $\rho_A \otimes \rho_B$ is given by

$$\rho_A \otimes \rho_B = \sum_{i\mu} a_i b_\mu |i\mu\rangle \langle i\mu| , \quad (3.9)$$

that is related to diagonal matrix. Using this fact, we simply obtain

$$\begin{aligned} \tilde{\rho}(\rho_A \otimes \rho_B) &= \sum_{i\mu} \sum_{j\nu} c(i\mu|j\nu) a_j b_\nu |i\mu\rangle \langle j\nu| , \\ (\rho_A \otimes \rho_B) \tilde{\rho} &= \sum_{i\mu} \sum_{j\nu} a_i b_\mu c(i\mu|j\nu) |i\mu\rangle \langle j\nu| . \end{aligned}$$

The commutator of operators $\tilde{\rho}$ and $\rho_A \otimes \rho_B$ has zero matrix elements, whence

$$(a_j b_\nu - a_i b_\mu) c(i\mu|j\nu) = 0 \quad (3.10)$$

for all values of labels. Under the precondition (ii) of theorem, it follows from (3.10) that off-diagonal elements $c(i\mu|j\nu)$ are all zero. Simplifying the notation to $c_{i\mu} \equiv c(i\mu|i\mu)$, both the sets $\{a_i\}$ and $\{b_\mu\}$ are marginal distributions of $\{c_{i\mu}\}$ due to (3.5). Then the statement of Lemma II.3 completes the proof. \square

Remark III.4. Let pair of qubits 'A' and 'B' be in the entangled pure state $|\Psi(\theta)\rangle = \cos\theta|00\rangle + \sin\theta|11\rangle$. Then $\tilde{\rho} = |\Psi(\theta)\rangle\langle\Psi(\theta)|$ is projector, and so its partial entropic sums are all zero. We also have $\rho_A = a_0|0\rangle\langle 0| + a_1|1\rangle\langle 1|$ and $\rho_B = b_0|0\rangle\langle 0| + b_1|1\rangle\langle 1|$, where $a_0 = b_0 = (\cos\theta)^2$ and $a_1 = b_1 = (\sin\theta)^2$. Except for $\cos 2\theta = 0$ and $\sin 2\theta = 0$, the operators $\tilde{\rho}$ and $\rho_A \otimes \rho_B$ are noncommuting. In this case, the eigenvalues of both ρ_A and ρ_B are strictly positive, whence $S_\alpha^{(k)}(\rho_A) > 0$ and $S_\alpha^{(k)}(\rho_B) > 0$.

Remark III.5. When $\cos 2\theta = 0$, the operators $\tilde{\rho}$ and $\rho_A \otimes \rho_B$ are commuting, but the inequalities (3.7) are still not satisfied. Indeed, we have $a_0 = b_0 = a_1 = b_1 = 1/2$, $S_\alpha^{(k)}(\rho_A) > 0$, and $S_\alpha^{(k)}(\rho_B) > 0$, while $S_\alpha^{(k')}(\tilde{\rho}) = 0$ for all k' as before. Here the precondition (ii) of Theorem III.3 is clearly violated. This example shows an insufficiency of the precondition (i) itself. The precondition (ii) is useful since only spectra of reduced operators are involved. These spectra may be known from the specification.

Another application of (2.5) is related to quantum measurements. A generalized measurement is described by "positive operator-valued measure" (POVM). Recall that POVM $\{M_j\}$ is a set of positive operators M_j satisfying [11, 14]

$$\sum_{j=1}^N M_j = \mathbb{1} , \quad (3.11)$$

where $\mathbb{1}$ is the identity in \mathcal{H} . Following [6], we consider operators of rank one, i.e. $M_j = |w_j\rangle\langle w_j|$. Let $\{q_i, |\psi_i\rangle\}$ be one of those ensembles that generates given density operator ϱ . Generally, the vectors $|w_j\rangle$ and $|\psi_i\rangle$ are neither normalized nor orthogonal. Let $P(\psi_i; w_j)$ denote the joint probability that the state $|\psi_i\rangle$ has been input and j -th outcome has been obtained. For normalized $|w_j\rangle$, the term $|\langle w_j | \psi_i \rangle|^2$ is the probability that $|\psi_i\rangle$ passes the "yes/no" test of being the state $|w_j\rangle$. Hence we obtain

$$P(\psi_i; w_j) = q_i |\langle w_j | \psi_i \rangle|^2 , \quad (3.12)$$

$$S_\alpha^{(k)}(\varrho) \leq H_\alpha^{(kN)}(P(\psi_i; w_j)) , \quad (3.13)$$

by using (2.5) with $r_{ij} = P(\psi_i; w_j)$ and $j = 1, \dots, N$. The above "yes/no" tests with pure states are easily realized in physical experiments. In many tasks of quantum information processing, a support and likewise spectrum sketch of unknown mixed state are certain *a priori*. (Recall that support of an operator is the subspace orthogonal to its kernel [11].) So the relation (3.13) may be applied in design of feasible schemes to estimate the entropic sums in practical specifications of such a kind. Here we consider a given support as the actual Hilbert space in which the decomposition (3.11) is taken. In principle, this issue might be a subject of separate research.

IV. INEQUALITIES FOR THE CASE $\alpha \in (0; 2]$

In this section we establish a desired upper bound on difference between two k -th entropic sums, when $\alpha \in (0; 2]$. In principle, the question can be recast as follows. For any two k -dimensional real vectors \mathbf{x} and \mathbf{y} , we define the function

$$F(\mathbf{x}, \mathbf{y}) := \sum_{i=1}^k \eta_\alpha(x_i) - \sum_{i=1}^k \eta_\alpha(y_i) . \quad (4.1)$$

To show the continuity and stability properties for partial entropic sums, we should obtain an upper bound on modulus of this function in the domain $\mathfrak{D}_\epsilon \subset \mathbb{R}^k \times \mathbb{R}^k$ specified by

$$\mathfrak{D}_\epsilon := \{(\mathbf{x}, \mathbf{y}) : x_i \geq 0, y_i \geq 0, G_{(k)}(\mathbf{x}) \leq 1, G_{(k)}(\mathbf{y}) \leq 1, G_{(k)}(\mathbf{x} - \mathbf{y}) \leq \epsilon\} . \quad (4.2)$$

Direct maximization of $|F(\mathbf{x}, \mathbf{y})|$ in \mathfrak{D}_ϵ seems to be difficult. Even if the optimization is made for $(\mathbf{x}, \mathbf{y}) \in \Delta_k \times \Delta_k$, when both the x_i 's and the y_i 's are summarized to 1, the task is very complicated. For instance, the function is neither convex nor concave. Any extensions of admissible domain hamper a solution. In the case $(\mathbf{x}, \mathbf{y}) \in \Delta_k \times \Delta_k$, the two complementary approaches are known.

The first method of estimating with restriction $\alpha \in [0; 2]$ is presented in [8]. This is generalization of that given for the von Neumann entropy in [12] (see proposition 1.8 therein). By relevant modifications, we obtain the following result.

Lemma IV.1. *If $\epsilon \leq \alpha^{1/(1-\alpha)}$ and $\alpha \in (0; 2]$ then in the domain \mathfrak{D}_ϵ it holds that*

$$|F(\mathbf{x}, \mathbf{y})| \leq \epsilon^\alpha \ln_\alpha(k+1) + \eta_\alpha(\epsilon) . \quad (4.3)$$

Proof. If $x, y \in [0; 1]$ and $|x - y| = t < 1$, then the maximum of $|\eta_\alpha(x) - \eta_\alpha(y)|$ is either $\eta_\alpha(t)$ or $\eta_\alpha(1 - t)$. When $\alpha \in (0; 2]$, the term $h(t) \equiv \eta_\alpha(t) - \eta_\alpha(1 - t)$ is positive for $t \in [0; 1/2]$, since $h(0) = h(1/2) = 0$ and $h''(t) \leq 0$ for $t \leq 1 - t$. So, we have

$$|\eta_\alpha(x) - \eta_\alpha(y)| \leq \eta_\alpha(t) . \quad (4.4)$$

When $t \leq \alpha^{1/(1-\alpha)}$, by monotonicity this bound holds for those points that $|x - y| \leq t$. Due to (4.4), the quantity $|F(\mathbf{x}, \mathbf{y})|$ does not exceed

$$\begin{aligned} \sum_{i=1}^k |\eta_\alpha(x_i) - \eta_\alpha(y_i)| &\leq \sum_{i=1}^k \eta_\alpha(\epsilon_i) = -\epsilon^\alpha \sum_{i=1}^k \frac{\epsilon_i^\alpha}{\epsilon^\alpha} \ln_\alpha\left(\frac{\epsilon_i}{\epsilon}\right) \\ &= -\epsilon^\alpha \sum_{i=1}^k \left\{ \left(\frac{\epsilon_i}{\epsilon}\right)^\alpha \ln_\alpha\left(\frac{\epsilon_i}{\epsilon}\right) + \frac{\epsilon_i}{\epsilon} \ln_\alpha(\epsilon) \right\} = \epsilon^\alpha \sum_{i=1}^k \eta_\alpha\left(\frac{\epsilon_i}{\epsilon}\right) + \eta_\alpha(\epsilon) \sum_{i=1}^k \frac{\epsilon_i}{\epsilon} , \end{aligned} \quad (4.5)$$

where we put $\epsilon_i \equiv |x_i - y_i|$ and use the identity $\ln_\alpha(xy) = \ln_\alpha(x) + x^{1-\alpha} \ln_\alpha(y)$. The last relations hold under the conditions $\alpha \in (0; 2]$ and $\sum_{i=1}^k \epsilon_i \leq \epsilon \leq \alpha^{1/(1-\alpha)}$. Using the upper bound from (2.8) and $\sum_{i=1}^k (\epsilon_i/\epsilon) \leq 1$, we finally obtain (4.3). \square

Remark IV.2. When $G_{(k)}(\mathbf{x} - \mathbf{y}) = \epsilon$, i.e. $\sum_{i=1}^k (\epsilon_i/\epsilon) = 1$, the set $\{\epsilon_i/\epsilon\}$ is a probability distribution supported on k points. Then the multiplier of ϵ^α in the right-hand side of (4.5) is the Tsallis entropy itself, whence

$$\sum_{i=1}^k \eta_\alpha(\epsilon_i/\epsilon) \leq \ln_\alpha(k) , \quad |F(\mathbf{x}, \mathbf{y})| \leq \epsilon^\alpha \ln_\alpha(k) + \eta_\alpha(\epsilon) . \quad (4.6)$$

In fact, just the last was given in [8]. But we must allow $G_{(k)}(\mathbf{x} - \mathbf{y}) < \epsilon$ for partial sums. Really, in the three quantities $G_{(k)}[\eta_\alpha(\mathbf{p})]$, $G_{(k)}[\eta_\alpha(\mathbf{q})]$, and $G_{(k)}(\mathbf{p} - \mathbf{q})$ the index of summation can run over three distinct k -subsets of the set $\{1, \dots, m\}$.

Remark IV.3. If $\alpha > 2$ then we have $h''(t) > 0$ and $h(t) < 0$ for $0 < t < 1/2$. It is for this reason that the case $\alpha > 2$ cannot be analyzed within the considered method.

Although the calculated bound is not sharp, it allows to establish the continuity of partial sums. We pose the statement just for the quantum case, because the case of classical distributions is simultaneously treated. This is possible due to (3.4) and the following helpful result.

Lemma IV.4. *Let function $x \mapsto f(x)$ be positive-valued. For all $\mathbf{p}, \mathbf{q} \in \mathbb{R}^m$, there holds*

$$|G_{(k)}[f(\mathbf{p})] - G_{(k)}[f(\mathbf{q})] + Q| \leq \left| \sum_{i \in \mathcal{C}} (f(p_i) - f(q_i)) + Q \right| , \quad (4.7)$$

where the Q is independent of these sums, the \mathcal{C} is a k -subset of the set $\{1, \dots, m\}$.

Proof. First, we introduce two k -subsets \mathcal{A}, \mathcal{B} of the set $\{1, \dots, m\}$ such that

$$G_{(k)}[f(\mathbf{p})] = \sum_{i \in \mathcal{A}} f(p_i) , \quad G_{(k)}[f(\mathbf{q})] = \sum_{j \in \mathcal{B}} f(q_j) . \quad (4.8)$$

Putting intersection $\mathcal{I} = \mathcal{A} \cap \mathcal{B}$, the term $(G_{(k)}[f(\mathbf{p})] - G_{(k)}[f(\mathbf{q})] + Q)$ is recast as

$$\sum_{i \in \mathcal{I}} (f(p_i) - f(q_i)) + \sum_{i \in (\mathcal{A} \setminus \mathcal{I})} f(p_i) - \sum_{j \in (\mathcal{B} \setminus \mathcal{I})} f(q_j) + Q . \quad (4.9)$$

The differences $(\mathcal{A} \setminus \mathcal{I})$ and $(\mathcal{B} \setminus \mathcal{I})$ are sets of equal cardinality. Without loss of generality, the value (4.9) can be assumed to be positive. Due to the definition of \mathcal{B} , we have $f(q_j) \geq f(q_i)$ for all $j \in \mathcal{B}$ and $i \notin \mathcal{B}$, including $j \in (\mathcal{B} \setminus \mathcal{I})$ and $i \in (\mathcal{A} \setminus \mathcal{I})$. Hence

$$-\sum_{j \in (\mathcal{B} \setminus \mathcal{I})} f(q_j) \leq -\sum_{i \in (\mathcal{A} \setminus \mathcal{I})} f(q_i) ,$$

and the claimed statement is obtained by replacing the former sum with the latter. Then we take $\mathcal{C} = \mathcal{A}$ in the inequality (4.7). \square

We are now able to prove the main result of this section. For two density operators $\boldsymbol{\rho}$ and $\boldsymbol{\varrho}$ on d -dimensional Hilbert space \mathcal{H} , we denote $p_i = \lambda_i^\downarrow(\boldsymbol{\rho})$ and $q_j = \lambda_j^\downarrow(\boldsymbol{\varrho})$ with $i, j \in \{1, \dots, d\}$. In this notation, we have

$$G_{(k)}[\mathbf{p} - \mathbf{q}] = G_{(k)}[\lambda^\downarrow(\boldsymbol{\rho}) - \lambda^\downarrow(\boldsymbol{\varrho})] \leq G_{(k)}[\lambda(\boldsymbol{\rho} - \boldsymbol{\varrho})] = \|\boldsymbol{\rho} - \boldsymbol{\varrho}\|_{(k)} , \quad (4.10)$$

The middle inequality is a particular case of theorem III.4.4 in [4]. By Lemma IV.4,

$$\left| S_\alpha^{(k)}(\boldsymbol{\rho}) - S_\alpha^{(k)}(\boldsymbol{\varrho}) \right| = \left| G_{(k)}[\eta_\alpha(\mathbf{p})] - G_{(k)}[\eta_\alpha(\mathbf{q})] \right| \leq \left| \sum_{i \in \mathcal{C}} (\eta_\alpha(p_i) - \eta_\alpha(q_i)) \right| \quad (4.11)$$

for some k -subset \mathcal{C} of the set $\{1, \dots, d\}$. Using $\epsilon := \sum_{i \in \mathcal{C}} |p_i - q_i|$ and $\varepsilon := \|\boldsymbol{\rho} - \boldsymbol{\varrho}\|_{(k)}$, the right-hand side of (4.11) does not exceed

$$\epsilon^\alpha \ln_\alpha(k+1) + \eta_\alpha(\epsilon) \leq \varepsilon^\alpha \ln_\alpha(k+1) + \eta_\alpha(\varepsilon) , \quad (4.12)$$

when $\varepsilon \leq \alpha^{1/(1-\alpha)}$. This follows from Lemma IV.1 and $\epsilon \leq G_{(k)}[\mathbf{p} - \mathbf{q}] \leq \varepsilon$ by (4.10). Thus, we have arrived at a conclusion that generalizes Fannes' inequality with respect to separate terms in the expression for entropy.

Theorem IV.5. *For given $\alpha \in (0; 2]$ and $k \in \{1, \dots, d\}$, if density operators $\boldsymbol{\rho}$ and $\boldsymbol{\varrho}$ satisfy the condition $\|\boldsymbol{\rho} - \boldsymbol{\varrho}\|_{(k)} = \varepsilon \leq \alpha^{1/(1-\alpha)}$ then*

$$\left| S_\alpha^{(k)}(\boldsymbol{\rho}) - S_\alpha^{(k)}(\boldsymbol{\varrho}) \right| \leq \varepsilon^\alpha \ln_\alpha(k+1) + \eta_\alpha(\varepsilon) . \quad (4.13)$$

V. INEQUALITIES FOR THE CASE $\alpha \in (2; +\infty)$

The second method of estimate with restriction $\alpha > 1$ is presented in [23]. Let \mathbf{p} and \mathbf{q} be m -dimensional probability vectors. The l_1 -distance (or Kolmogorov distance) between them is defined as [11]

$$D(\mathbf{p}, \mathbf{q}) := \frac{1}{2} \sum_{i=1}^m |p_i - q_i| \equiv \frac{1}{2} G_{(m)}(\mathbf{p} - \mathbf{q}) . \quad (5.1)$$

Using probabilistic coupling techniques, the following statement has been proved [23]. For $\alpha > 1$, there holds

$$\left| H_\alpha^{(m)}(\mathbf{p}) - H_\alpha^{(m)}(\mathbf{q}) \right| \leq \delta^\alpha \ln_\alpha(m-1) + H_\alpha(\delta, 1-\delta) , \quad (5.2)$$

where $\delta = D(\mathbf{p}, \mathbf{q})$ and $H_\alpha(\delta, 1-\delta) \equiv \eta_\alpha(\delta) + \eta_\alpha(1-\delta)$ is the binary Tsallis entropy [23]. The normalization conditions $\sum_{i=1}^m p_i = \sum_{i=1}^m q_i = 1$ are crucial for use of probabilistic coupling. Nevertheless, we can utilize the result (5.2) in obtaining estimates for partial sums of Tsallis entropy.

Remark V.1. The right-hand side of (5.2) can be rewritten as $g(\delta, m-1)$, where we put

$$g(\delta, n) := \delta^\alpha \ln_\alpha(n) + H_\alpha(\delta, 1-\delta) = (1-\alpha)^{-1} (n^{1-\alpha} \delta^\alpha + (1-\delta)^\alpha - 1) . \quad (5.3)$$

By definition, we have $\eta_\alpha(\delta) \leq g(\delta, n)$. As a function of δ at fixed n , the $g(\delta, n)$ monotonically increases in the range $0 < \delta < n/(n+1)$. Indeed, for $\alpha > 1$ the condition $\partial g / \partial \delta > 0$ leads to $(\delta/n) < (1-\delta)$.

Remark V.2. If $|B| \leq C$ and $A > 0$ then $|A+B| \leq C$ implies $A \leq C + |B|$ inevitably.

Lemma V.3. *If $\epsilon \leq \min \{ \alpha^{1/(1-\alpha)}, (k+1)/(k+2) \}$ and $\alpha > 1$ then in the domain \mathfrak{D}_ϵ it holds that*

$$|F(\mathbf{x}, \mathbf{y})| \leq g(\epsilon, k+1) + \eta_\alpha(\epsilon) . \quad (5.4)$$

Proof. With no loss of generality, the quantity $F(\mathbf{x}, \mathbf{y})$ can be meant as positive. Consider the two cases.

(i) Components of \mathbf{x} and \mathbf{y} obey $u + \sum_{i=1}^k x_i = \sum_{i=1}^k y_i = 1 - v$, where $u, v \geq 0$. For $m = k+2$, we build two m -dimensional probability vectors $\mathbf{p} = (x_1, \dots, x_k, u, v)$ and $\mathbf{q} = (y_1, \dots, y_k, 0, v)$.

(ii) Components of \mathbf{x} and \mathbf{y} obey $u + \sum_{i=1}^k y_i = \sum_{i=1}^k x_i = 1 - v$, where $u, v \geq 0$. For $m = k+2$, we build two m -dimensional probability vectors $\mathbf{p} = (x_1, \dots, x_k, 0, v)$ and $\mathbf{q} = (y_1, \dots, y_k, u, v)$.

After cancellation of $\eta_\alpha(v)$, for both the cases we can write

$$H_\alpha^{(m)}(\mathbf{p}) - H_\alpha^{(m)}(\mathbf{q}) = F(\mathbf{x}, \mathbf{y}) \pm \eta_\alpha(u) , \quad (5.5)$$

where the sign '+' is put in the case (i), the sign '-' is put in the case (ii). In both the cases, we also have

$$u = \left| \sum_{i=1}^k x_i - \sum_{i=1}^k y_i \right| \leq \sum_{i=1}^k |x_i - y_i| \leq G_{(k)}(\mathbf{x} - \mathbf{y}) = \epsilon ,$$

whence $2D(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^k |x_i - y_i| + u \leq 2\epsilon$. So we can take the result (5.2) with $\delta \leq \epsilon$ and $m = k + 2$. When $\epsilon \leq (k + 1)/(k + 2)$, we have $g(\delta, k + 1) \leq g(\epsilon, k + 1)$ and

$$|F(\mathbf{x}, \mathbf{y}) + B| \leq g(\epsilon, k + 1) .$$

Here we use (5.5) and $B = \pm \eta_\alpha(u)$. If $u \leq \epsilon \leq \min \{\alpha^{1/(1-\alpha)}, (k + 1)/(k + 2)\}$ then $|B| = \eta_\alpha(u) \leq \eta_\alpha(\epsilon) \leq g(\epsilon, k + 1)$, and the implication of Remark V.2 completes the proof. \square

It must be stressed that the two ranges $\alpha \in (0; 2]$ and $\alpha \in (1; +\infty)$, in which the two estimates (4.3) and (5.4) are respectively valid, have the joint interval $(1; 2]$. Here the bound (5.4) is weaker, and we shall use it for $\alpha \in (2; +\infty)$ only. For small ϵ , the difference between these bounds is small though. Similar to the inequality (4.13), the following result is immediately obtained from (5.4).

Theorem V.4. *For given $\alpha \in (2; +\infty)$ and $k \in \{1, \dots, d\}$, if density operators ρ and ϱ satisfy the condition $\|\rho - \varrho\|_{(k)} = \varepsilon \leq \min \{\alpha^{1/(1-\alpha)}, (k + 1)/(k + 2)\}$ then*

$$\left| S_\alpha^{(k)}(\rho) - S_\alpha^{(k)}(\varrho) \right| \leq \varepsilon^\alpha \ln_\alpha(k + 1) + \eta_\alpha(\varepsilon) + H_\alpha(\varepsilon, 1 - \varepsilon) . \quad (5.6)$$

Remark V.5. The inequalities (4.13) and (5.6) are expressed in terms of the distances $\|\rho - \varrho\|_{(k)}$ that enjoy similar properties to the standard trace distance [17]. These measures are closely related to Uhlmann's partial fidelities. In general, the fidelity concept provides a natural extension of notion of transition probability to mixed quantum states [21]. The k -fidelity between ρ and ϱ is defined as [22]

$$F_k(\rho, \varrho) := \sum_{j=k+1}^d \sigma_j^\downarrow(\sqrt{\rho}\sqrt{\varrho}) .$$

It has been shown [16] that $\|\rho - \varrho\|_{(k)} \leq 2(1 - F_k(\rho, \varrho))$. By replacing ε with $\varepsilon' := 2(1 - F_k(\rho, \varrho))$, we obtain the reformulations of (4.13) and (5.6) in terms of the partial fidelities under the corresponding conditions on ε' .

Theorems IV.5 and V.4 together fully cover the range $\alpha \in (0; +\infty)$. Note that both the bounds explicitly depend only on k and not on the dimension of \mathcal{H} . They establish that all the partial sums of the Tsallis entropy are continuous with respect to the trace norm. Indeed, for $k = 1, \dots, d$, we have $\|\rho - \varrho\|_{(k)} \leq \|\rho - \varrho\|_{(d)} \equiv \text{tr}|\rho - \varrho|$ by the definition (3.1). The inequalities (4.13) and (5.6) are useful, since previous extensions of Fannes' inequality dealt with the Tsallis entropy as a whole. However, by itself a continuity of some functional does not imply the same property for separate terms of the functional (see, for example, the ratio of (2.10) to (2.11)). As it is shown above, the partial entropic sums are also continuous. So the Tsallis entropy enjoys a continuity property that is more subtle in character.

Finally, we discuss a stability of partial entropic sums. In order for a measure to be experimentally robust, it is necessary to put the stability criterion proposed by Lesche [10]. This criterion is posed as follows [1, 23]. Let Φ be a state functional on the probability simplex Δ_m . We assume the set $\{\Phi(\mathbf{p}) : \mathbf{p} \in \Delta_m\}$ of positive numbers to be bounded above by the least bound Φ_M . The stability means that for every $\delta > 0$, there exists $\varepsilon > 0$ such that $|\Phi(\mathbf{p}) - \Phi(\mathbf{q})| \cdot \Phi_M^{-1} \leq \delta$ whenever $G_{(m)}(\mathbf{p} - \mathbf{q}) \leq \varepsilon$. Since $G_{(k)}(\mathbf{p} - \mathbf{q}) \leq G_{(m)}(\mathbf{p} - \mathbf{q})$, we have

$$\left| H_\alpha^{(k)}(\mathbf{p}) - H_\alpha^{(k)}(\mathbf{q}) \right| \leq g(\varepsilon, k + 1) + \eta_\alpha(\varepsilon) \quad (5.7)$$

provided that $G_{(m)}(\mathbf{p} - \mathbf{q}) = \varepsilon \leq \varepsilon_0$, where $\varepsilon_0 = \min \{\alpha^{1/(1-\alpha)}, (k + 1)/(k + 2)\}$. The relations (5.7) and (2.8) do enjoy the stability property for partial entropic sums. Indeed, to each $\xi \in (0; \varepsilon_0)$ we assign

$$\delta(\xi) = (g(\xi, k + 1) + \eta_\alpha(\xi)) \ln_\alpha(k)^{-1} ,$$

so that the inequality

$$\left| H_\alpha^{(k)}(\mathbf{p}) - H_\alpha^{(k)}(\mathbf{q}) \right| \left(\max H_\alpha^{(k)} \right)^{-1} \leq \delta(\xi) \quad (5.8)$$

is herewith stated for all $G_{(m)}(\mathbf{p} - \mathbf{q}) \leq \xi$. The function $\delta(\xi)$ vanishes at the point $\xi = 0$ and monotonically increases with ξ for $\xi \in (0; \varepsilon_0)$ (see Remarks II.1 and V.1). So, one defines some one-to-one correspondence between the intervals $[0; \varepsilon_0]$ and $[0; \delta(\varepsilon_0)]$. Therefore, for each $\delta \in (0; \delta(\varepsilon_0)]$ we have $\xi(\delta) > 0$ such that the stability condition (5.8) does hold whenever $G_{(m)}(\mathbf{p} - \mathbf{q}) \leq \xi(\delta)$. This is a particular example to the fact that Lesche's stability property is formally equivalent to uniform continuity [1, 23].

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